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Extended self-similarity in turbulent flows

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We report on the existence of a hitherto undetected form of self-similarity, which we call extended self-similarity (ESS). ESS holds at high as well as at low Reynolds number, and it is characterized by the same scaling exponents of the velocity differences of fully developed turbulence.

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There is experimental and numerical evidence that the energy spectrum $E(k)$ of fully developed turbulence exhibits a well-defined scaling law, close to $k^{-5/3}$, as predicted by the Kolmogorov theory [1]. More generally let us consider the probability density function (PDF) of the velocity increments $\Delta V(r) \equiv V(x+r) - V(x)$, where $V(x+r)$ and $V(r)$ are velocities along the x axis at two points separated by a distance r and $\eta \ll r \ll L$, L being the integral scale of motion and η the dissipation scale. The Kolmogorov theory predicts $\langle \Delta V(r)^n \rangle \approx r^{\zeta(n)}$ with $\zeta(n) = n/3$ in the fully developed regime, and, in particular, one can rigorously prove, see below [2], that $\zeta(3) = 1$. The Kolmogorov predictions are based upon the assumptions that the statistical properties of the velocity field are locally homogeneous and isotropic and that there exists a constant-energy cascade from large to small scales. In this case one can assume that the PDF of $\Delta V(r)$ depends only on r and the average rate of energy dissipation ϵ. Extensive experimental and numerical investigations have highlighted slight deviations from the Kolmogorov prediction $\zeta(n) = n/3$, $n \neq 3$, which are due to the strong intermittent character of the energy dissipation [3–5]. However, in addition to the quantitative predictions of the Kolmogorov theory for the scaling exponents $\zeta(n)$, the existence of universal scaling laws for $\langle \Delta V(r)^n \rangle$ has been verified for a variety of different turbulent flows at high Reynolds number. This is equivalent to saying that the statistical properties of the velocity field are self-similar within the inertial range, i.e., for $\eta \ll r \ll L$, at high Reynolds numbers. The aim of this Rapid Communication is to show that the statistical properties of turbulence could be self-similar also at low Reynolds number, and moreover they could be characterized by the same set of scaling exponents $\zeta(n)$ of the fully developed regime.

Let us start by remembering that, within the assumptions of local homogeneity and isotropy, from the Navier-Stokes equations one can deduce [2] the following relation:

$$\langle \Delta V(r)^3 \rangle = -\tfrac{4}{5}\epsilon r + 6\nu \frac{d}{dr} \langle \Delta V(r)^2 \rangle , \tag{1}$$

where ν is the kinematic viscosity and $\langle\ \rangle$ stands for average over the PDF of $\Delta V(r)$. For $r \gg \eta \equiv \nu^{3/4}\epsilon^{-1/4}$ the second term of the right-hand side (rhs) in Eq. (1) can be neglected, showing that $\zeta(3) = 1$, as previously mentioned. The existence of an inertial range in data analysis is usually deduced by probing the scaling of $\langle \Delta V(r)^3 \rangle$ versus r: the range of scales where such a scaling law is verified indicates the "inertial range." Because of (1), within the inertial range, one can readily write

$$\begin{aligned} \langle |\Delta V(r)|^n \rangle &= A_n |\langle \Delta V(r)^3 \rangle|^{\zeta(n)} \\ &= B_n \langle |\Delta V(r)|^3 \rangle^{\zeta(n)} , \end{aligned} \tag{2}$$

where A_n and B_n are two different sets of constants and the relation $|\langle \Delta V(r)^3 \rangle| \approx \langle |\Delta V(r)|^3 \rangle$, which cannot be trivially deduced by the Navier Stokes equations, is verified experimentally. Our claim is that Eq. (2), with the same scaling exponents of fully developed turbulence, is valid not only in the fully developed regime but also at moderate low Reynolds number, i.e., even if no inertial range (according to the usual definition given above) is established. Moreover, it will be shown that the range of scales for which scaling (2) is much larger than the inertial range (when it exists), i.e., self-similarity of the velocity field extends far beyond the usual inertial range deep into the dissipation range.

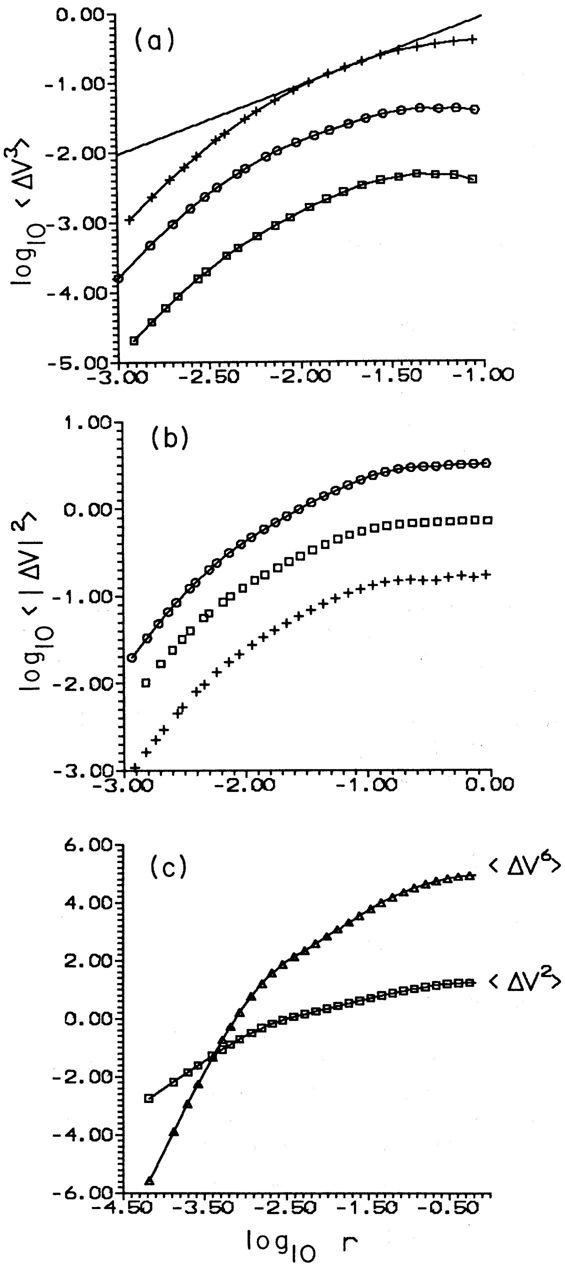

FIG. 1. (a) Log-log plot of $|\langle \Delta V(r)^3 \rangle|$ vs r for three different values of Reynolds number in the case of flow past a cylinder: 6000 (squares), 22 500 (circles), and 47 000 (crosses). Only for Reynolds number 47 000 can a rather small inertial range (indicated by the line with slope 1) be observed. (b) Same as (a) for the second-order structure functions, namely, $\langle \Delta V(r)^2 \rangle$. The corresponding Reynolds numbers are 6000 (squares), 22 500 (circles), and 47 000 (crosses). (c) Log-log plot of $\langle \Delta V(r)^6 \rangle$ and $\langle \Delta V(r)^2 \rangle$ vs r for the case of the jet flow.

Our claims are supported by a number of experimental and numerical results, some of which are described elsewhere [6]. Here we confine our attention to a set of experimental data obtained by hot-wire measurements of the velocity field in a wind tunnel [6]. Turbulence is generated either by a flow past a cylinder or by a jet. The cylinder has a diameter of 6 cm and we considered Reynolds number $\mathrm{Re} \equiv UL/\nu = 6000$, 22 500, and 47 000, where U is the incoming flow velocity in the wind against the cylinder, $\nu = 0.156 \times 10^{-4}\ \mathrm{m^2\,sec^{-1}}$ is the kinematic viscosity of air, and L is the diameter of the cylinder. The measurements reported here are taken at about $20L$ down flow. The results discussed below do not depend on this distance. The jet has a diameter of 12 cm and the flow velocity at the exit of the jet is of the order of 35 m/s. Thus the Reynolds number is on the order of 300 000.

In Figs. 1(a) and 1(b) we show $\langle \Delta V(r)^3 \rangle$ and $\langle \Delta V(r)^2 \rangle$, respectively, for the three different values of Re in the case of the flow past a cylinder. Only at $\mathrm{Re} = 47\,000$ is a (very questionable) scaling law proportional to r observed for $\langle \Delta V(r)^3 \rangle$ in a very small interval of r. A value of $\zeta(2)$ can be estimated from Fig. 1(b) using the same range of scales. This yields $\zeta(2) \approx 0.7$. In the same way the exponents $\zeta(n)$ for $n = 1,4,6,8$ can be estimated. The corresponding results (see Fig. 2) are not very different from the values reported for turbulent flows at much higher Reynolds numbers [3]. We expect, however, a quite large statistical error due to the limited range of scales of the inertial range. In Fig. 1(c) we show $\langle \Delta V(r)^6 \rangle$ and $\langle \Delta V(r)^2 \rangle$ for the case of jet flow at Re of order 300 000. In this case a clear scaling is observed at least for one decade in r with slope $\zeta(6) = 1.78$ and $\zeta(2) = 0.7$, respectively.

Next in Fig. 3(a) we plot $\langle \Delta V(r)^2 \rangle$ versus $\langle |\Delta V(r)|^3 \rangle$ for $\mathrm{Re} = 6000$ and 47 000 in the case of the flow past a cylinder ($\mathrm{Re} = 22\,500$ is not displayed in order to clarify the figure). A striking and much wider scaling range is

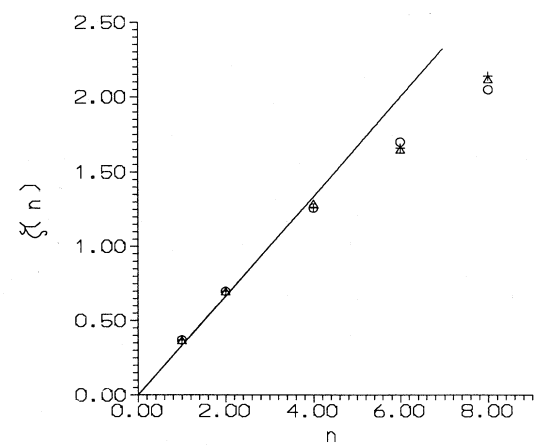

FIG. 2. Scaling exponents $\zeta(n)$ computed for $n = 1,2,4,6,8$. Triangles represent the values obtained by measuring the scaling of $\langle \Delta V(r)^n \rangle$ against r at $\mathrm{Re} = 47\,000$; circles and crosses represent the scaling exponents computed using ESS (see text) for $\mathrm{Re} = 6000$ and 47 000, respectively. The straight line represents the Kolmogorov scaling $n/3$.

displayed in Fig. 3(a) [a similar result is obtained by plotting $\langle \Delta V(r)^2 \rangle$ versus $\langle \Delta V(r)^3 \rangle$]. In Fig. 3(b) we plot $\langle \Delta v(r)^6 \rangle$ versus $\langle \Delta V(r)^2 \rangle$ for the case of jet flow, and once again we observe a much wider range of scaling almost down to the Kolmogorov scale, which is in this case of the order of 10^{-5} m. The results shown so far are neither predicted by Kolmogorov theory nor by any other theory developed in the last few years to explain intermittency effects in scaling laws of fully developed turbulence. In the case of the flow past a cylinder a fit of the data at Re=6000 yields $\zeta(2)=0.700\pm0.005$. The same fit applied to the full data set (the three different values of Re) yields $\zeta(2)=0.701$. Note that there is statistical evidence that $\zeta(2)$ differs from the "naive" value $\frac{2}{3}$. Results similar to those shown in Fig. 3 hold for $\langle |\Delta V(r)|^n \rangle$ and $n>3$. From these scaling laws we have extracted the exponents $\zeta(n)$ at Re=6000, and in Fig. 2 we compare the different sets of exponents; the agreement is excellent. Equivalent results (i.e., same exponents) have been obtained from an experiment on grid turbulence and in the case of jet flow.

The previously described results allow us to argue that self-similarity as expressed by Eq. (2) is somehow more fundamental than the self-similar scaling with respect to r usually observed at very high Reynolds numbers. In the following we shall refer to Eq. (2) as extended self-similarity (ESS) of the velocity field. It is reasonable, although still speculative, to predict that ESS holds for many other turbulent flows such as those arising in magnetohydrodynamics, thermal convection, two-dimensional turbulence, quasigeostrophic turbulence. In all cases ESS could produce far reaching consequences both from an experimental and a numerical point of view. For instance, an accurate estimate for the scaling exponents $\zeta(n)$ could be obtained at low Reynolds number for which direct numerical simulations are already able to provide quite accurate data sets.

In order to further investigate the validity of ESS, we proceed as follows. Let us assume that Eq. (2) is valid for $r \geq \eta$. Then for the second-order structure function we have

$$\langle \Delta V(r)^2 \rangle = A_2 |\langle \Delta V(r)^3 \rangle|^{\alpha} .$$

Using ESS in Eq. (1) we obtain

$$-S(r) = -\tfrac{4}{5}\epsilon r + 6\nu A_2 \frac{d}{dr} S(r)^{\alpha} , \qquad (3)$$

where $S(r)=|\langle \Delta V(r)^3 \rangle|$. We have integrated Eq. (3) with the parameter corresponding to Fig. 1 for Re=22 500. In this case we have estimated $A_2=2.818$ $(\mathrm{m/sec})^{2-3\alpha}$ and $\alpha=0.700$. By using Fig. 1 we have estimated, in the inertial range, $\epsilon \equiv \frac{5}{4}[|\langle \Delta V(r)^3 \rangle|/r] = 1.7\ \mathrm{m^2\,sec^{-3}}$, corresponding to a value of the Kolmogorov scale $\eta=0.2$ mm. In Fig. 4 we compare the numerical solution against the experimental values of $S(r)$. The agreement is excellent, taking into account that no adjustable parameter has been used. Similar results hold for different Reynolds number and for the case of jet flow. This result clearly confirms what has been already shown

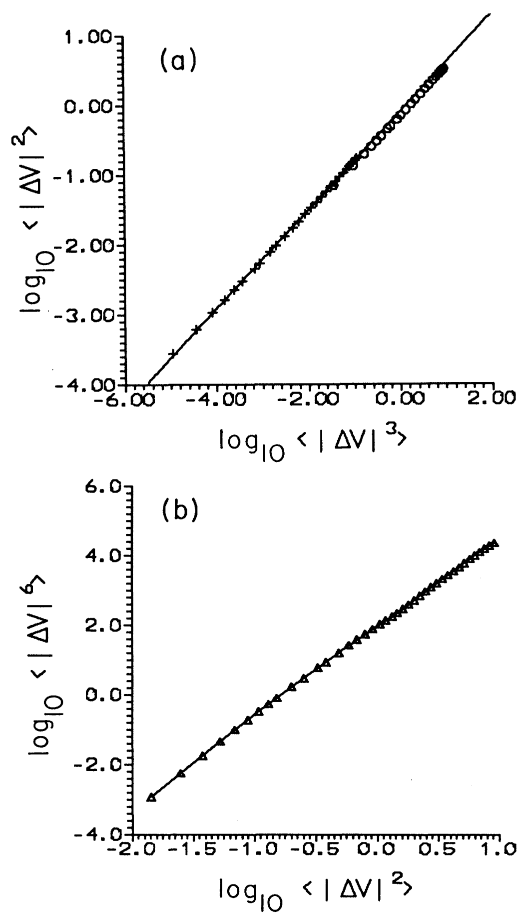

FIG. 3. Log-log plot of $\langle |\Delta V(r)^2| \rangle$ vs $\langle |\Delta V(r)|^3 \rangle$ for the case of flow past a cylinder at Re=6000 (squares) and 47 000 (crosses). The line represents the best fit done using the data for Re=6000 and Eq. (2). The best fit is given by $\langle |\Delta V(r)|^2 \rangle = B_2 \langle |\Delta V(r)|^3 \rangle^{0.700}$. (b) Log-log plot of $\langle \Delta V(r)^6 \rangle$ vs $\langle \Delta V(r)^2 \rangle$ for the case of jet flow.

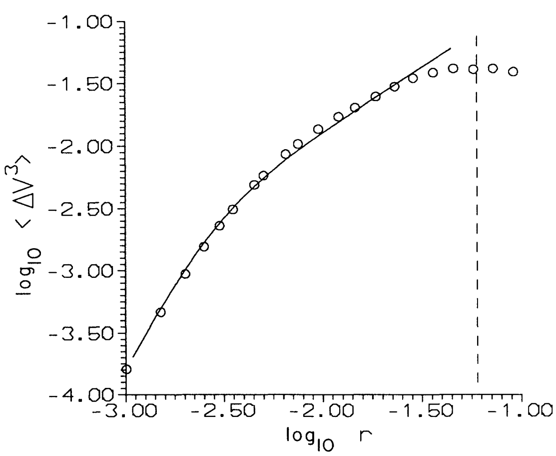

FIG. 4. Numerical solution of Eq. (3) (solid line) compared against experimental data (circles) for Re=22 500 in the case of flow past a cylinder. The dashed line indicates the size of the cylinder (6 cm diameter). Note that the agreement between Eq. (3) and experimental data is lost only at very large scales.

in Fig. 3, i.e., that the inertial range, defined by the scaling of the second-order structure function against $S(r)$, extends much further than the "naive" inertial range, defined as the scaling of $S(r)$ against r. This is equivalent to saying that the bending of the structure functions for small r does not imply a lack of self-similarity in the dissipation range, contrary to common belief.

At a scale $r \approx \eta$ (never achieved in our experiments) the scaling (2) should be violated. At that scale new phenomena could occur, as recently pointed out [7–9]. Perhaps a reinterpretation in terms of Eq. (2) of available numerical data could clarify the dynamics in the far dissipation range. This is a matter for future research.

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